The Philippine Statistician
1990, Vol. 39, Nos. 1-2 & 3-4,
pages 52-60

STORAGE SYSTEMS WITH INPUT AS A CORRELATED BIRTH AND DEATH PROCESS

by

K.S. Rao, J. Lakshminarayana

and

V. Papayya Sastry
Department of Statistics,
Andhra University
Visakhaptnam 530 003 (A.P.) INDIA

Abstract

In this paper an infinite capacity storage system in which the input rate follows correlated birth and death process is analysed by considering the constant demand rate. The almost null recurrent distribution of the content is obtained. An approximation for the joint distribution of the content and input rate during the period of excessive over supply is also obtained by Gaussian diffusion approximation.

1. INTRODUCTION

Various storage models have been developed and analysed with appropriate modifications to suit practical situations. In general, storage processes are considered with infinite and finite capacities having random input process. Many authors analysed storage models in which the content is increased and decreased alternatively, and the increase and decrease are independently and identically distributed {[3], [9] and [1]}. [2] and [8] have considered the finite capacity storage systems in which the input rate is a two state Markov process in continuous time and the demand rate was a general function of the content. [4] and [10] analysed the storage systems, in which the input rate follows birth and death process. In these models the increase and decrease of the input process are independently distributed.

In many of the storage systems like regulated input systems in industries and urban areas, the increase and decrease are interdependent, in order to have a feasible operating policy. For this sort of systems, an attempt is made in this article by considering a storage model in which the input rate follows a correlated birth and death process, where the increase and decrease in the input process are correlated random variables following a bivariate Poisson of the form [6],

$$P(X_1=x_1, X_2=x_2\} = e^{-(\lambda+\mu-\epsilon)t}$$

$$\sum_{j=0}^{\min(x_1,x_2)} \frac{(\epsilon t)^j (\lambda t-\epsilon t)^{x_1-j} (\mu t-\epsilon t)^{x_2-j}}{j!\ (x_4-j)!\ (x_2-j)!} \qquad \begin{matrix} 0 \le x_1,\ x_2 \le \infty \\ ,\ \epsilon < \min(\lambda,\ \mu) \\ 0 < \lambda\ ,\ \mu. \end{matrix}$$

Then, the increase and decrease are Poisson variates with λ and μ as mean increase and mean decrease rate having a mean dependence rate ϵ.

This distribution was obtained from three independent Poisson variates (viz. u,v and θ) with the relation $X_1 = \mu + \theta$ and $X_2 = v + \theta$ If we let X_1 be the increase and X_2 be the decrease then θ is the dependence function. Let $X(t)$ and $R(t)$ be the content and input rate respectively at time t $\{t \ge 0\}$ where $X(t)$ is defined over $[0,\infty]$. With the above structure, the transition probabilities may be approximated as

$$P\ \{R[t+\delta(t)-R(t)=j\,|\,R(t)=m\}= \begin{cases} [\lambda(m)-\epsilon]\delta t+O(\delta t) \text{ for } j=1 \\ [\mu(m)-\epsilon(m)]\delta t+O(\delta t) \text{ for } j=-1 \\ 1-[\lambda(m)-\mu(m)-2\epsilon(m)]\delta t+O(\delta +) \\ \qquad \text{for } j=0 \\ O(\delta t) \text{ otherwise.} \end{cases}$$

There is an equivalence for the transition probabilities and the bivariate Poisson process mentioned earlier.

Assuming the system is not empty, let the rate of change in $X(t)$ be

$$\frac{d[X(t)]}{dt} = R(t) - D[X(t)],$$

when the input rate $R(t)$ and the demand rate $D[X(t)]$ is a continuous function of the content. Suitable modifications are made when the system is empty. Then $\{X(t),R(t)\}$ is a continuous time Markov Process. The above system turns out to be a Herbert system when the dependence parameter reduces to zero.

NULL RECURRENT DISTRIBUTION

Consider an infinite capacity storage system with the above assumptions. Assuming the stationarity of the process, the parameters of the model are

$$\lambda(m) = \lambda \qquad (m = 0,1,2,\ldots.)$$
$$\mu(m) = \mu \qquad (m = 1,2,3,\ldots.)$$
$$\epsilon(m) = \epsilon \qquad (m = 0,1,2,\ldots.)$$

and the constant demand rate is α. Then the stationary content distribution exists if $(\lambda-\epsilon) < \alpha(\lambda-\mu)$, [7].

Let $P_m(x)$ be the probability that $X(t) \le x$ and $R(t) = m$ when $X(0) = x_0$ and $R(0) = r_0$ for $m = 0,1,2,\ldots.$

Then the difference - diffrential equations of the model are

$$(-\alpha)\frac{dP_0(x)}{dx} = (\lambda-\epsilon)P_0(x) + (\mu-\epsilon)P_1(x)$$

$$(m-\alpha)\frac{dP_m(x)}{dx} = -(\lambda-\epsilon)P_{m-1}(x) - (\lambda+\mu-2\epsilon)P_m(x)$$

$$+ (\mu-\epsilon)P_{m+1}(x), \quad x \ge 0, \; m=1,2,..k$$

and $$(m-\alpha)\frac{dP_m(x)}{dx} = (\lambda-\epsilon)P_{m-1}(x) + (\lambda+\mu-2\epsilon)P_m(x)$$

$$+(\mu-\epsilon)Pm+1(x), \quad x>0 \text{ and } m=k+1, k+2,\ldots$$

Consider the Laplace Transform of $P_m(x)$

$$\pi_m(s) = \int_0^\infty e^{-sx} P_m(x)\, dx.$$

Then we have

$$(-\alpha)\ s\ \pi_0(s) + \alpha P_0(0) = -(\lambda-\epsilon)\pi_0(s) + (\mu-\epsilon)\pi(s)$$

$$(m-\alpha)\ s\ \pi_m(s)\ (\alpha-m)P_m(0) = (\lambda-\epsilon)\pi_m(s) - (\lambda+\mu-2\epsilon)\ \pi_m(s) + (\mu-\epsilon)\pi_{m+1}(s)\ ,\ m=1,2,\dots,k$$

$$(m-\alpha)\ s\pi_m(s) = (\lambda-\epsilon)\pi_{m-1}(s) - (\lambda+\mu-2\epsilon)\pi_m(s) + (\mu-\epsilon)\ \pi_{m+1}(s),\ m = k+1,\ k+2,\dots \tag{1}$$

Solving these equations we get

$$\pi_{k+m}(s) = \left(\frac{\lambda-\varepsilon}{\mu-\epsilon}\right)^{m/2} \left[\frac{J_{v+k+m}(q)}{J_{v+k}(q)}\right] \pi_k(s) \tag{2}$$

where $J_v(q)$ is the Bessel function of order v and argument q, $v=[\lambda+\mu-2\epsilon)/s] - \alpha$, and $q=2\sqrt{[(\lambda-\epsilon)\ (\mu-\epsilon)/s]}$.

From equations (1) and (2)

$$\pi_0(s) \doteq \frac{\sum_{r=0}^{k} (\alpha-r)\left(\frac{\mu-\epsilon}{-\epsilon}\right)^{r/2}\left(\frac{J_{v+r}(q)}{J_v(q)}\right) P_r(0)}{(\mu-\epsilon) - \sqrt{((-\epsilon)(\mu-\epsilon))}(J_{v-1}(q)/J_v(q)} \tag{3}$$

$P_r(0)$ for $r = 1,2,\dots k$ can be obtained by taking the limit of $s\pi_0(s)$ at $s = 0$.

$$\frac{\mu-\lambda}{\mu-\epsilon} = \frac{\sum_{r=0}^{k} (\alpha-r)\left(\frac{\mu-\epsilon}{\lambda-\epsilon}\right)^{\frac{r}{2}} P_r(0) \lim_{s\to 0} \left(\frac{J_{v+r}(q)}{J_v(q)}\right)}{\sqrt{[(\lambda-\epsilon)\ (\mu-\epsilon)]}\lim_{s\to 0} \frac{d}{ds}\left(\frac{J_{v-1}(q)}{J_v(q)}\right)} \tag{4}$$

Using the Bessel function approximations found in [5] and after simplication we have

$$\sum_{r=0}^{k} (\alpha-r)\, P_r(0) = \left(\frac{\lambda-\epsilon}{\mu\ \lambda}\right), \quad r = 0,1,2,\ldots,k \tag{5}$$

From these set of k equations, equation (1) can be solved for $\pi_m(s)$ (m =0,1,2...).

For obtaining the most null - recurrent distribution

let $P^{p}_{m}(x) = \lim_{t\to\infty} P\{X^{(p)}(t) \le x,\ R(t)$

$= m \mid X(0)=x_o,\ R(0) = r_o\};\ m = 0,1,2,\ldots.$

where $P^{p}\pi(s)$ is the lim of $P^{p}_{m}(x)$, $X^{(p)}(t) = (1-p)\,X(t)$ and $p = (\lambda-\epsilon)/\alpha(\mu-\lambda)$. From equations (3), (4) and (5) and after some simplification we have

$$\lim_{p\to 1} s(1-p)\pi\,[s(1-p) = \frac{1-\left(\dfrac{\lambda-\epsilon}{\mu-\epsilon}\right)}{1+(\mu-\lambda)\, s\, f\left(p-\epsilon,\ \lambda-\epsilon,\ \dfrac{-\epsilon}{a(\mu-\lambda)}\right)} \tag{6}$$

where

$$f(\mu,\lambda,\alpha) = \frac{\lambda\,2\alpha+1}{4(\mu-)^{2}} + \frac{(\lambda+\mu)(3\alpha+1)+(6\lambda(\alpha+1)^{2}}{6(\mu-\lambda)^{3}}$$

$$+ \frac{3(\lambda+\mu)^{2}-32(\lambda-\mu)(\alpha+1)}{8(\mu-\lambda)^{4}}$$

$$+ \frac{16\lambda\,\mu(\lambda^{2}+\mu^{2})(\alpha+1) + 16\lambda^{2}\mu^{2}(8\alpha^{2}+6\alpha+1) - (\alpha+\mu)^{4}}{8(\mu-\lambda)^{6}}$$

From equation (6) we have

$$\lim_{p \to 1^-} s(1-p)\pi_m [s(1-p)] = \left(\frac{\lambda \ \epsilon}{\mu-\lambda}\right)^m \lim_{p \to 1^-} \pi_0^{(p)}(s), \ m=1,2,... \quad (7)$$

Thus

$$P_m^{(p)} = \frac{\frac{\mu-\lambda}{\mu-\epsilon} \left(\frac{\lambda-\epsilon}{\mu-\epsilon}\right)^m [1-e^{-\alpha x}]}{(\mu-\lambda) \ f\left(\mu-\epsilon, \lambda-\epsilon, \frac{\lambda-\epsilon}{\alpha(\mu-\lambda)}\right)}, \ x > 0, \ m=0,1...,$$

DIFFUSION APPROXIMATION

Since the transient behaviour of the joint process $\{X(t), R(t)\}$ is complicated, a diffusion approximation (limit) of the joint process $\{X((t), R(t))$ when there is excessive over supply is obtained using the heuristic argument in [4]. Assume that at time t=0, the system is in excessive over supply. Following [8], a parameter N is included in the set of the parameters. The parameters of the system under consideration are

$$\lambda_N(m) = N\lambda \qquad (m = 0,1,2,....)$$

$$\mu_N(m) = N\mu \qquad (m = 0,1,2,.....)$$

$$\epsilon_N(m) = N\epsilon \qquad (m = 0,1,2.....)$$

with the condition that $\lambda < \mu$.

For the constant demand rate

$$D\ [X_N(t)] = N\ \alpha_N$$

where $\alpha_N = \alpha + O\ (N^{-1/2})$

consider the joint process

$$[X'_N(t), R'_N(t)] = \{[X_N(t) - N\ M(t)]\ N,$$

$$[R_N(t) - N\ m(m(t)]\ N\}.$$

With the above set up

$$\dot{m}(t) = (\lambda-\mu) + m(0)$$

$$\mu(t) = 1/2[(\lambda-\mu)\, t^2] + m(0)t + \alpha t + \mu(0), \quad M(0) > \alpha$$

Then the stochastic behavior of the system is

$$\frac{\partial \phi}{\partial t} = s_1 \frac{\partial \phi}{\partial s_2} - \frac{1}{2} (\lambda + \mu - 2\epsilon)(s_2)^2 \phi$$

with the initial condition

$$\phi\,(s_1, s_2; 0) = e^{(is\, X_1\,(0)\, +\, is\, R_2(0)\,)}$$

where ϕ is the characteristics function of the joint process $\{X'(t), R'(t)\}$.

Solving the above partial differential equation for ϕ in the same lines of [4] we obtain

$$\phi(s_1, s_2, t) = \exp\left[is_1\, a_1\,(t) + is_2 a_2(t) - \frac{(s_1)^2}{2} b_1(t) - s_2\, b_{12}(t) - \frac{(s_2)^2}{2} b_2(t)\right],$$

where,

$$a_1(t) = R'\,(0)\,t + X'(0)$$

$$a_2(t) = R'(0)$$

$$b_1(t) = \frac{1}{3}\,(\lambda + \mu - 2\,\epsilon)\,t^3$$

$$b_{12}(t) = \frac{1}{3}\,(\lambda + \mu - 2\,\epsilon)\,t^2$$

$$b_2(t) = (\lambda + \mu - 2\,\epsilon)\,t.$$

Thus ϕ is the characteristic function of a Gaussian process. Therefore, the joint process $\{X_N(t), R_N(t)\}$ is asymptotically normal with mean vector

$$[N/2\ (\lambda-\mu)t^2 - N\ \alpha\ t + R(0)\ t + X(0);\ N(\lambda-\mu)\ t + R(0)\ ]$$

and variance - covariance matrix

$$\begin{bmatrix} \frac{1}{3} N\ (\lambda+\mu-2\epsilon)\ t^3 & \frac{1}{3} N(\ +\mu-2\epsilon)t^2 \\ \frac{1}{2}\ N\ (\ +\mu-2\epsilon)\ t^2 & N(\lambda+\ \mu-2\epsilon)\ t \end{bmatrix}$$

The correlation coefficient between X(t) and R(t) is $\sqrt{3}/2$. Therefore the distribution of the content in the storage at t is approximately normally distributed.

REFERENCES

[1] Brockwell, P.J. (1972) A storage model in which the net growth rate is a Markov chain *Journal of Appl. Prob.* **9**.

[2] Erickson, R. V. (1972) Stationary measures for a class of storage models driven by a Markov chain. *'Ann. Math. Stat.* **43**.

[3] Gaver, D. P. and Miller, R. G. Jr. (1962). 'Limiting distribution for some storage problems '*Studies in Applied Probability and Management Science.*

[4] Herbert, H. G. (1974) 'Some approximate results for storage systems with continuous inputs'. *Journal of App. Prob.* **11**.

[5] Jeffers, H. and B. S. (1948). *Mathematical Physics,* Cambridge University Press.

[6] Johnson and Kotz (1972). *Discrete Distributions.* John Wiley, N.Y.

[7] Loynes, R. M. (1962). 'The stability of a queue with non-independent interarrival and service times Proc *Camp. Philos.* Soc. **58**.

[8] McNeil, D. R. (1972). A 'simple model for a dam in continuous time with Markovian input Z. *Wahrscheinlickhhkeitsth* **21**.

[9] Miller, R. G. (1963) Continuous time storage processes with random linear inputs and outputs. *Journal of Math. Mech.* **12**.

[10] Vandoorn, E. A., Jagfrsa, A. and Dewit J. T. (1988). A fluid reservoir regulated by a birth-death process *Stoch. Models* **4**.